PHYSICAL REVIEW E VOLUME 51, NUMBER 3 MARCH 1995

Suppression of higher harmonics at noise induced resonances

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(Received 24 August 1994)

We consider the generation of higher harmonics in periodically driven noisy nonlinear systems. Recent numerical studies of higher harmonics in such systems have shown so-called *noise induced resonances* that manifest themselves in a strong suppression of higher harmonics at certain values of the noise level. A theory for this peculiar phenomenon is presented, unmasking the universal character of these resonances and their widespread occurrence.

PACS number(s): 02.50.−r, 05.40.+j

I. INTRODUCTION

Much attention has been paid in the past years to the response of nonlinear noisy systems to periodic signals [1–13]. It has been demonstrated that noise can actually amplify periodic signals in bistable filters due to stochastic resonance. Harmonic mixing in a noisy nonlinear system was discussed in great detail in Ref. [14]. The impact of noise on the generation of higher harmonics has become of interest only very recently. The response of a periodically driven noisy system to an additional small harmonic signal was studied in [15]. The dependence of the intensity and phases of higher harmonics on the noise strength was studied in detail for a model for absorptive optical bistability [16]. Stochastic resonance was reported also in [17] for a nonlinear mixing process in the presence of fluctuations. The impact of noise on the distortion of nonperiodic signals—also an effect due to the generation of higher harmonics—was studied by analog simulations in [18]. Higher harmonics generation in a periodically driven bistable system has also been addressed recently for weak noise and small frequencies within a two-state approach in Ref. [20]. One of the most surprising findings, however, is the resonancelike suppression of higher harmonics at certain values of the noise strength accompanied by phase jumps of π, an effect that has been termed *noise induced resonance* (NIR) [16]. This phenomenon has been discovered on the basis of a numerical study in asymmetric bistable [16] as well as in asymmetric stable systems [19]. In this paper, we present a theory for NIR that is applicable for continuous and discrete systems—stable or multistable.

In Sec. II we derive an expression for the amplitudes of higher harmonics in periodically driven, noisy nonlinear systems in terms of the cumulants of the undriven process. We derive general conditions under which noise induced resonances occur. In Sec. III we apply our general result to a symmetric and an asymmetric monostable system. In Sec. IV we consider an asymmetric two-state system and a symmetric three-state system.

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II. GENERAL THEORY

We consider a nonlinear noisy system with a periodic input $A\sin\Omega t$ and an output, described by the Langevin equation

$$\dot{x} = f(x) + \xi(t) + A\sin\Omega t, \tag{1}$$

with white Gaussian noise $\xi(t)$, i.e.,

$$\langle\xi(t)\xi(t')\rangle = 2D\delta(t-t'), \quad \langle\xi(t)\rangle = 0. \tag{2}$$

The output x and time t as well as all parameters are considered to be dimensionless. The corresponding probability density approaches for large times an asymptotic probability density [10], which is given for small driving frequencies Ω by

$$P_{\rm as}(x,t) = \frac{1}{Z}\exp\left(-\frac{U(x)}{D} - \frac{Ax\sin\Omega t}{D}\right), \tag{3}$$

with $U'(x) = -f(x)$. We restrict ourselves here to bounded systems, i.e., $f(x)$ is such that the process $x(t)$ approaches a stationary process without the external signal $A\sin\Omega t$. The partition function Z is given by

$$Z = Z(\alpha(t)) = \int_{-\infty}^{\infty}\exp\left(-\frac{U(x)}{D} - \alpha(t)x\right)dx, \tag{4}$$

with

$$\alpha(t) = \frac{A}{D}\sin\Omega t. \tag{5}$$

The spectral density of a periodically driven stochastic process consists of a continuous Lorentz-like background $S_b(\omega)$ plus δ spikes $S_s(\omega)$ at the multiples of the driving frequency [10]. The δ spikes correspond to the periodicity of the time-averaged correlation function $\langle\langle x(t+\tau)x(t)\rangle\rangle_t$ for large time separations τ

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$$\langle\langle x(t+\tau)x(t)\rangle\rangle_t \overset{\tau\to\infty}{\to} \langle\langle x(t+\tau)\rangle\langle x(t)\rangle\rangle_t = \sum_{n=-\infty}^{\infty} |c_n|^2 \exp(in\Omega\tau),$$

$$S_s(\omega)= 4\pi \sum_{n=-\infty}^{\infty} |c_n|^2\delta(\omega - n\Omega), \quad (6)$$

where c_n are the complex-valued Fourier coefficients of the time-dependent mean value $\langle x(t)\rangle = \int_{-\infty}^{\infty} xP_{\rm as}(x,t)dx$. In order to obtain the weight $|c_n|^2$ of the harmonics in the power spectrum, we have to calculate the Fourier series of the mean value $\langle x(t)\rangle$ by using the adiabatic probability density Eqs. (3) and (4)

$$\langle x(t)\rangle = -\frac{\partial}{\partial\alpha}\ln Z(\alpha(t)). \quad (7)$$

In terms of the unperturbed ($A = 0$) partition function Z_0, the partition function $Z(\alpha(t))$ can be written as

$$Z(\alpha(t)) = Z_0\Phi(\alpha(t)), \quad (8)$$

with the characteristic function of the unperturbed process $\Phi(s) = \langle\exp(-sx)\rangle_0$. The mean value $\langle x(t)\rangle$ is then given by the derivative of the cumulant generating function, i.e.,

$$\langle x(t)\rangle = -\frac{\partial}{\partial\alpha}\ln\Phi(\alpha(t)) = -\frac{\partial}{\partial\alpha}\sum_{n=1}^{\infty}\frac{K_n}{n!}(-\alpha(t))^n, \quad (9)$$

where K_n are the cumulants of the unperturbed process. Reordering the sum in (9) in terms of a Fourier series one obtains with (5)

$$\langle x(t)\rangle = \sum_{l=0}^{\infty}\frac{K_{2l+1}}{(l!)^2}\left(\frac{A}{2D}\right)^{2l} +2\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{A}{2D}\right)^{n}\cos\left(n\Omega t + n\frac{\pi}{2}\right) \times\sum_{l=0}^{\infty}\frac{K_{n+2l+1}}{l!(n+l)!}\left(\frac{A}{2D}\right)^{2l}. \quad (10)$$

The explicit summation of (10) requires the knowledge of *all* cumulants of the unperturbed process, given in terms of the moments $M_n \equiv \langle x^n\rangle$ by the determinant [21]

$$K_n = (-1)^{n-1}\begin{vmatrix} M_1 & 1 & 0 & 0 & 0 & \cdots \\ M_2 & M_1 & 1 & 0 & 0 & \cdots \\ M_3 & M_2 & \binom{2}{1}M_1 & 1 & 0 & \cdots \\ M_4 & M_3 & \binom{3}{1}M_2 & \binom{3}{2}M_1 & 1 & \cdots \\ M_5 & M_4 & \binom{4}{1}M_3 & \binom{4}{2}M_2 & \binom{4}{3}M_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n .$$

Restricting ourselves to leading order contributions of each harmonic, Eq. (10) becomes

$$\langle x(t)\rangle \approx K_1 + 2\sum_{n=1}^{\infty}\left(\frac{1}{n!}\right)^2\left(\frac{A}{2D}\right)^n K_{n+1} \times\cos\left(n\Omega t + n\frac{\pi}{2}\right), \quad (11)$$

where the intensity of each harmonic is characterized by a single cumulant. The weight of the overtones in the spectral density $|c_n|^2$ can then be written in terms of the cumulants of the undriven process

$$\gamma_n \equiv \frac{4\pi n!^2|c_n|^2}{A^{2n}} = \frac{4\pi}{(2D)^{2n}}\left[\sum_{l=0}^{\infty}\left(\frac{A}{2D}\right)^{2l}\frac{K_{n+2l+1}}{l!(n+l)!}\right]^2 \quad (12)$$

$$\approx \frac{4\pi}{(2D)^{2n}}\left(\frac{K_{n+1}}{n!}\right)^2. \quad (13)$$

Noise induced resonances, i.e., the zeros of the Fourier coefficients c_n, are therefore connected with zeros of cumulants. The basic harmonic ($n = 1$), given by the second cumulant, is positive, implying the nonexistence of NIR's at the basic harmonic at weak signals. All the other cumulants can have a positive and a negative sign yielding the possibility of noise induced resonances. For larger A/D the next-to-leading order contributions in (12) become relevant [i.e., Eq. (12) has to be applied], resulting in a shift or removal of NIR.

Equations (10)–(13) are the main results and are discussed for specific examples below.

III. CONTINUOUS SYSTEMS

In this section we study noise induced resonances in a symmetric monostable system, characterized by the Langevin equation (1) (in dimensionless units) with

$$U(x) = \frac{1}{2}|x|^\gamma + \frac{1}{4}x^4 \quad (14)$$

and $\gamma \leq 2$. To show the occurrence of NIR's, we analyze the cumulants of the unperturbed process at small and large noise strengths separately.

Computing the stationary moments at weak noise, the x^4 term in Eq. (14) can be neglected, yielding

$$K_2 = \langle x^2\rangle = \frac{\Gamma(3/\gamma)}{\Gamma(1/\gamma)}(2D)^{2/\gamma},$$

$$K_4 = \langle x^4\rangle - 3\langle x^2\rangle^2 = (2D)^{4/\gamma}\frac{1}{\Gamma^2(1/\gamma)}\left[\Gamma(5/\gamma)\Gamma(1/\gamma) - 3\Gamma^2(3/\gamma)\right]. \quad (15)$$

For $\gamma < 2$, K_4 is positive, while for $\gamma = 2$, i.e., for a Gaussian distribution, the fourth cumulant K_4 vanishes by definition.

For large noise D, the $|x|^\gamma$ term in (14) can be neglected, yielding for the stationary cumulants

$$K_2 = \frac{\Gamma(3/4)}{\Gamma(1/4)}(4D)^{1/2},$$

$$K_4 = 4D\frac{1}{\Gamma^2(1/4)}\left[\Gamma(5/4)\Gamma(1/4) - 3\Gamma^2(3/4)\right] < 0. \quad (16)$$

For $\gamma < 2$, the fourth cumulant and therefore—for weak modulation—the amplitude of the third harmonic have at least one zero. Numerical computation of the stationary cumulants yields exactly one zero, i.e., one NIR in the third harmonic (see Fig. 1). For $\gamma = 2$, i.e., for a parabolic potential minimum, the cumulants, starting out from zero at weak noise, increase monotononically without a zero.

The situation is different for asymmetric systems with a parabolic minimum, e.g.,

$$U_{\rm as}(x) = \frac{1}{2}x^2 + \frac{1}{4}x^4 - ax. \quad (17)$$

At large noise, the term $-ax$ causing the asymmetry can be neglected and the cumulants are dominated by the quartic term in Eq. (17), i.e., are given by Eq. (16). For weak noise, an asymptotic evaluation of the first four moments yields for the cumulants

$$K_2 = \frac{D}{\omega^2}\left(1 - \frac{3D}{\omega^4} + \frac{36x_0^2D}{\omega^6} + O(D^2)\right),$$
$$K_3 = -\frac{6x_0D^2}{\omega^6}\left(1 - \frac{21D}{\omega^4} + \frac{144x_0^2D}{\omega^6} + O(D^2)\right),$$
$$K_4 = -\frac{6D^3}{\omega^8}\left(1 - \frac{18x_0^2}{\omega^2} + O(D)\right), \quad (18)$$

where x_0 denotes the location of the minimum of the potential (17) and $\omega^2 = 1 + 3x_0^2$ the second derivative of the potential at x_0. For $x_0^2 > 1/15$ the fourth cumulant changes its sign with increasing noise strength leading to at least one NIR in the third harmonic. The location of the precise position of the NIR requires the full calculation of the cumulants.

In Fig. 2 the intensity of the third harmonic (13) is shown as a function of the noise strength D. We find one NIR, i.e., one zero of the cumulant, which is shifted towards larger values of the noise strength for increasing asymmetry a. We also want to mention the remarkably good agreement between the full numerical Fokker-Planck analysis of the system with the potential (17) in [19] and our analytical prediction based on the cumulants. For $c = 0.5$, $A = 0.01$, and $\Omega = 0.01$, the NIR in the third harmonic has been found in [19] at $D = 0.109$. The zero of the cumulant K_4 agrees up to the third digit with this value.

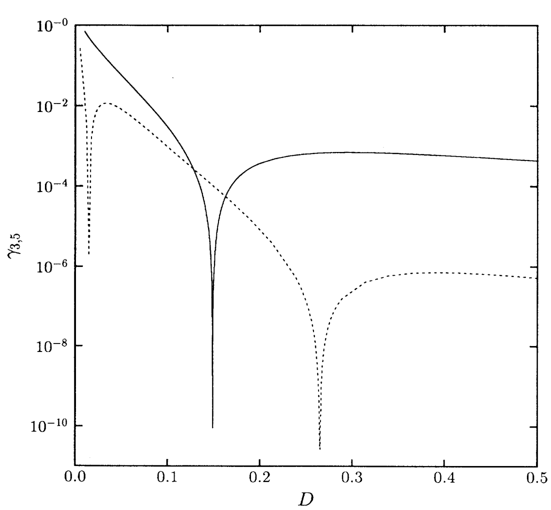

FIG. 1. The scaled weights of the third (solid line) and the fifth (dashed line) overtone γ_3 and γ_5 in the symmetric system are shown as a function of the noise intensity.

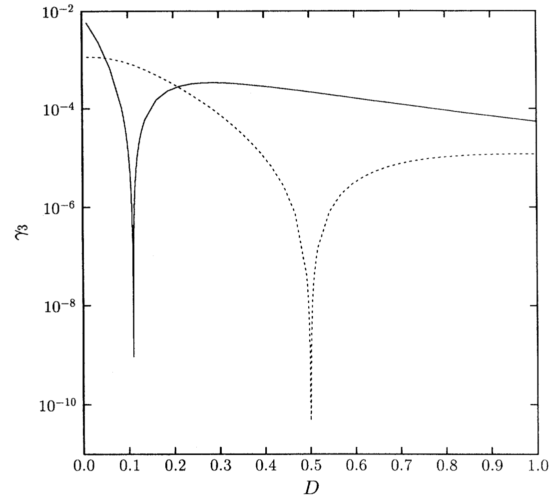

FIG. 2. The scaled weight of the third overtone γ_3 is shown for the asymmetric monostable model at $a = 0.5$ (solid line) and $a = 1$ (dashed line) as a function of the noise intensity.

IV. DISCRETE SYSTEMS

We now want to apply our general theory to discrete systems, i.e., systems with only a finite number of discrete states x_n. The most simple of such systems is a two-state system with states at $-x_0$ and x_0. Denoting the transition rates from $\pm x_0$ to $\mp x_0$ by $r_\mp(t)$ and the probability for the system being at time t at $\pm x_0$ by $p_\pm(t)$, the two-state dynamics is described by the master equation

$$\dot{p_\pm}(t) = r_\pm(t)p_\mp(t) - r_\mp(t)p_\pm(t). \quad (19)$$

Without external periodic fields ($r_\pm$ are time independent), the stationary probabilities $p_\pm$ are given by $p_\pm = r_\pm/(r_+ + r_-)$. The cumulants of the stationary density are readily obtained as

$$K_1 = (p_+ - p_-)x_0,$$
$$K_2 = 4p_+p_-x_0^2,$$
$$K_3 = -8p_+p_-(p_+ - p_-)x_0^3,$$
$$K_4 = -2(3(p_+ - p_-)^2 - 4(p_+ - p_-) + 1)x_0^4. \quad (20)$$

The second cumulant is only zero in the trivial cases $p_+ = 1$ or 0. The third cumulant is identically zero for the symmetric two-state system ($p_+ = 1/2$); otherwise it does not change sign by varying p_+ between 0 and 1. The fourth cumulant has a nontrivial zero at

$$p_\pm = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right), \quad (21)$$

i.e., for an asymmetric two-state system. Using the same notation as in [19], i.e., $r_+ = r_- \exp(2F/D)$, the NIR of the third harmonic is located at

$$D = \frac{2F}{|\ln(p_+/p_-)|}. \tag{22}$$

The numerically found NIR's for this system in [19] do not compare as well with the analytical values above (10% deviation), since the signal strengths used in the simulations in [19] are too large for the approximation (13) to hold.

Discrete symmetric systems also show NIR's. As an example we consider the symmetric three-state system. It is characterized by the three states x_0, $-x_0$, and 0. The stationary probability of the system being in either of the two states $\pm x_0$ is given by p, while the probability of being at $x = 0$ is correspondingly $1 - 2p$. The stationary cumulants are given by

$$K_2 = 2px_0^2, \ K_4 = 2px_0^4(1 - 6p). \tag{23}$$

The fourth cumulant has a nontrivial zero at $p = 1/6$, i.e., the third harmonic shows a noise induced resonance.

V. CONCLUSIONS

We have derived an analytical expression for the amplitudes of higher harmonics in nonlinear noisy systems driven by a periodic signal, valid for small driving frequencies. Conditions have been given under which a noise induced resonance will occur. We have shown that these conditions can be met by continuous as well as discrete, stable, and multistable systems.

ACKNOWLEDGMENTS

P.J. wishes to thank the Deutsche Forschungsgemeinschaft for financial support.

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